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## AN ANALYTICAL SOLUTION OF BIOT'S PROBLEM.

BY TSURUICHI HAYASHI.

In H. Laurent's *Traité d'Analyse*, tome 5, 1890, p. 110, we find the following problem due to Biot: Find a plane curve, such that all the luminous rays emanating from a fixed point, after two reflections on the curve, return to the fixed point. Laurent's solution is very simple, applying the common law of reflection, but it is wrong. Prof. M. Fijiwara has given a true solution to the problem in the *Tohoku Mathematical Journal*, volume 2, 1912, p. 149. I shall here give an analytical solution.

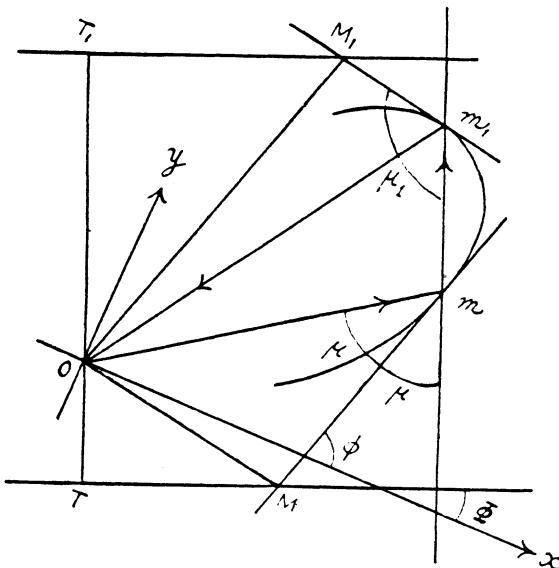


FIG. 1.

Let  $O$  be the source of light, and let  $Om m_1 O$  be the path of a ray. Let  $\mu$  be the angle which  $Om$  or  $mm_1$  makes with the tangent  $mM$  at  $m$ , and let  $\mu_1$  be the angle which  $mm_1$  or  $m_1O$  makes with the tangent  $m_1M_1$  at  $m_1$ . Drop from  $O$  perpendiculars  $OM$ ,  $OM_1$  on  $mM$ ,  $m_1M_1$  respectively. Then  $M$ ,  $M_1$  lie on the pedal of the required curve.

Take  $O$  as origin and  $Ox$ ,  $Oy$  as rectangular coördinate axes. Then the rectangular coördinates  $(x, y)$  of the point  $m$  are connected with the polar tangential coördinates  $(p, \varphi)$  of the same point by the relation

$$x \sin \varphi - y \cos \varphi = p,$$

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$\mu$  being the perpendicular  $OM$ , and  $\varphi$  being the angle between  $Mm$  and  $Ox$ . This relation is the equation to the tangent  $Mm$ . Hence this equation and that got by differentiating it with respect to  $\varphi$ , i.e.,

$$x \cos \varphi + y \sin \varphi = dp/d\varphi, = p' \text{ say,}$$

give the rectangular coördinates  $(x, y)$  of the point  $m$  in terms of the polar tangential coördinates  $(p, \varphi)$ . Thus

$$x = p \sin \varphi + p' \cos \varphi,$$

$$y = -p \cos \varphi + p' \sin \varphi.$$

The equation of  $OM$  is

$$x \cos \varphi + y \sin \varphi = 0.$$

Therefore the rectangular coördinates  $(X, Y)$  of the point  $M$  are

$$X = p \sin \varphi, \quad Y = -p \cos \varphi.$$

Hence the tangent to the pedal of the required curve, i.e., the locus of  $M$ , at  $M$  has the direction given by

$$\frac{dY}{dX} = -\frac{d(p \cos \varphi)}{d(p \sin \varphi)} = -\frac{p' \cos \varphi - p \sin \varphi}{p' \sin \varphi + p \cos \varphi}, = \tan \Phi \text{ say.}$$

Now by the relation

$$x \cos \varphi + y \sin \varphi = p',$$

$mM$  is equal to  $p'$ , so that

$$\tan \mu = p/p'.$$

Hence the angular coefficient of  $mm_1$  is given by

$$\tan(\varphi + \mu) = \frac{p' \sin \varphi + p \cos \varphi}{p' \cos \varphi - p \sin \varphi}.$$

Therefore

$$\frac{\pi}{2} + \Phi = \varphi + \mu,$$

i.e.,  $mm_1$  and  $TM$  make right angles.

Similarly  $m_1m$  and  $T_1M_1$ , tangent to the pedal at  $M_1$ , make right angles. Therefore  $TM$  and  $T_1M_1$  are parallel, and the perpendiculars  $OT$  and  $OT_1$  dropped from  $O$  on  $TM$  and  $T_1M_1$  respectively lie on one and the same straight line. Denote the length of  $OT$  by  $P$ , so that the polar tangential coördinates of the point  $M$  are  $(P, \Phi)$ , while its rectangular coördinates are  $(X, Y)$ .

The equation to  $mm_1$ , regarded as passing through  $m$ , is

$$\eta + p \cos \varphi - p' \sin \varphi = \tan(\varphi + \mu) \cdot (\xi - p \sin \varphi - p' \cos \varphi),$$

$\xi, \eta$  being current coördinates. But

$$\begin{aligned}
 -\tan(\varphi + \mu) \cdot (p \sin \varphi + p' \cos \varphi) - p \cos \varphi + p' \sin \varphi \\
 = -\frac{2pp'}{p' \cos \varphi - p \sin \varphi} \\
 = -2pp'(p^2 + p'^2)^{-\frac{1}{2}}(\cos \mu \cos \varphi - \sin \mu \sin \varphi)^{-1} \\
 = -2pp'(p^2 + p'^2)^{-\frac{1}{2}}\{\cos(\varphi + \mu)\}^{-1} \\
 = 2pp'(p^2 + p'^2)^{-\frac{1}{2}}(\sin \Phi)^{-1}.
 \end{aligned}$$

Similarly, from the equation to  $mm_1$ , regarded as passing through  $m_1$ , we have

$$\begin{aligned}
 -\tan(\varphi_1 + \mu_1) \cdot (p_1 \sin \varphi_1 + p_1' \cos \varphi_1) - p_1 \cos \varphi_1 + p_1' \sin \varphi_1 \\
 = 2p_1p_1'(p_1^2 + p_1'^2)^{-\frac{1}{2}}(\sin \Phi_1)^{-1}.
 \end{aligned}$$

These two expressions must be equal, since they come from the equations to the same straight line  $mm_1$ . Hence

$$pp'(p^2 + p'^2)^{-\frac{1}{2}}(\sin \Phi)^{-1} = p_1p_1'(p_1^2 + p_1'^2)^{-\frac{1}{2}}(\sin \Phi_1)^{-1}.$$

But

$$\Phi_1 = \pi + \Phi.$$

Therefore

$$pp'(p^2 + p'^2)^{-\frac{1}{2}} + p_1p_1'(p_1^2 + p_1'^2)^{-\frac{1}{2}} = 0.$$

Now by a well-known theorem,  $OM$  is the mean proportional between  $OT$  and  $OM$ .\* Hence

$$P = p^2(p^2 + p'^2)^{-\frac{1}{2}}.$$

Therefore

$$\frac{dP}{d\Phi} = \frac{p^3p' + 2pp'^3 - p^2p'p''}{(p^2 + p'^2)^{3/2}} \cdot \frac{d\varphi}{d\Phi}.$$

But from the relation

$$\varphi + \mu = \varphi + \tan^{-1} \frac{p}{p'} = \frac{\pi}{2} + \Phi,$$

we have

$$\frac{d\varphi}{d\Phi} = \frac{p^2 + p'^2}{p^2 + 2p'^2 - pp''}.$$

Therefore

$$\frac{dP}{d\Phi} = \frac{pp'}{(p^2 + p'^2)^{1/2}}.$$

Similarly

$$\frac{dP_1}{d\Phi_1} = \frac{p_1p_1'}{(p_1^2 + p_1'^2)^{1/2}}.$$

\* See, e.g., Williamson's Differential Calculus, 1892, p. 228.

But the sum of the right-hand members is equal to zero as has been shown above. Therefore

$$\frac{dP}{d\Phi} + \frac{dP_1}{d\Phi_1} = 0,$$

i.e.,

$$P'(\Phi) + P'(\Phi + \pi) = 0.$$

Integrating with respect to  $\Phi$ ,

$$P(\Phi) + P(\Phi + \pi) = \text{const.}$$

Hence *the pedal of the required curve is a curve of constant breadth.*

The converse can be similarly treated. *Singular solutions* are got by putting

$$\tan(\varphi + \mu) = 0 \quad \text{or} \quad = \infty,$$

i.e.,

$$p' \sin \varphi + p \cos \varphi = 0, \quad \text{or} \quad p' \cos \varphi - p \sin \varphi = 0,$$

i.e.,

$$p = \frac{\text{const.}}{\sin \varphi}, \quad \text{or} \quad p = \frac{\text{const.}}{\cos \varphi}.$$

Therefore the oval included by two confocal parabolas having the same axis, but in opposite senses, is the required curve.

SENDAI, JAPAN,  
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